Construction of Finite Iteration Runge-Kutta Methods
with a One-Point Spectrum for Solving Stiff Initial Value Problems

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2015-06-01
Kolloquium des Instituts für Mathematik
Universität Kassel
PWR – schematics
PWR – example network
Modelling of thermo-hydraulic processes

- based on conservation of mass, energy, momentum for liquid and vapor
- spatial discretisation by some finite volume approach

leading to an Initial Value Problem

\[ y' = f(t, y), \quad y(t_0) = y_0 \]

with \( f : \mathbb{R} \times \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}^n \).

Example ODE:

\[
X_m' = \left[ \frac{m_v}{m_l + m_v} \right]'
\]

where

\[ m_v = (1 - X_m) \cdot M, \quad m_l = X_m \cdot M \]

\[ M = \frac{V_{\text{const}}}{X_m \nu_v + (1 - X_m) \nu_l} \]
Properties and demands
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- discontinuities and dimension changes may arise in $f$
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- one-step RK-like methods
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- actual order \( p = \min(p_{max}, m) \)
Properties and demands

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  Small order methods
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- often $f \in C^m(\Omega, \mathbb{R}^n)$
- only for small $m$
- one-step RK-like methods
- small order methods
- Initial values and error tolerance often lead to stiff IVPs
Properties and demands

- Discontinuities and dimension changes may arise in $f$.
- Often $f \in C^m(\Omega, \mathbb{R}^n)$ only for small $m$.

One-step RK-like methods

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Implicit RK-like methods
Properties and demands

- discontinuities and dimension changes may arise in $f$
  
  often $f \in C^m(\Omega, \mathbb{R}^n)$
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  the Jacobian $\frac{\partial f}{\partial y}(\cdot, \cdot) =: f^{(1)}(\cdot, \cdot)$
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- $\exists D \subset \text{spec}(f^{(1)}(\cdot, \cdot))$:
- $D \neq \emptyset \land \Re(\lambda) < 0, \ |\lambda| \gg |\mu| \ \forall \lambda \in D, \ \forall \mu \in \text{spec}(f^{(1)}(\cdot, \cdot)) \setminus D$
Properties and demands

- Discontinuities and dimension changes may arise in $f$
- Often $f \in C^m(\Omega, \mathbb{R}^n)$ only for small $m$
- Initial values and error tolerance often lead to stiff IVPs
- The Jacobian $\frac{\partial f}{\partial y} (\cdot, \cdot) =: f^{(1)}(\cdot, \cdot)$ is considered as the (main) source of stiffness
- Rosenbrock/W-methods
Properties and demands

- discontinuities and dimension changes may arise in $f$
- Jacobian is sparse $n \in [10^3, 10^4]$ usually
- often $f \in C^m(\Omega, \mathbb{R}^n)$ only for small $m$
- one-step RK-like methods
- small order methods
- Initial values and error tolerance often lead to stiff IVPs
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Jacobian pattern, example
Extrapolation based on linearly implicit Euler

\[ y_1 - y_0 = hf(t_1, y_1) \]  
(implicit Euler)

make autonomous system (add \( t' = 1 \)), linearize at \((t_0, y_0)\) treating \( t \)-part explicitly

\[ (I - hJ)(y_1 - y_0) = hf(t_0, y_0) + h^2 \frac{\partial f_0}{\partial t} \]  
(linearly implicit Euler)

\[ J \approx \frac{\partial f_0}{\partial y} \]  
(in principle arbitrary \( \rightarrow \) W-method)
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Extrapolation for \( \{n_j\} = \{1, 2, 3\} \):

\[ T_{j,k+1} = T_{j,k} + \frac{T_{j,k} - T_{j-1,k}}{n_j/n_{j-k} - 1} \]

(Neville-Aitken)
Extrapolation based on linearly implicit Euler

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Extrapolation for \( \{n_j\} = \{1, 2, 3\} \):
Stabilities and order reduction

**Dahlquist**

\[ y' = \lambda y, \quad \Re(\lambda) < 0, \quad y(t_0) = y_0 \]

- **solution:** \[ y(t_0 + h) = y_0 e^{\lambda h} \]
- **approximation:** \[ y_1 = R(z)y_0, \quad z = h\lambda, \quad R(z) \text{ stability function at } z \]}
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stability region: \[ S = \{ z \in \mathbb{C} \mid |R(z)| \leq 1 \} \]

\( A(\alpha) \)-stable: \[ S_\alpha = \{ z \in \mathbb{C} \mid \arg(-z) \leq \alpha \} \subset S \]

\( A \)-stable: \[ \alpha = 90^\circ \]

\( L \)-stable: \( A \)-stable and \[ R(\infty) := \lim_{z \to \infty} R(z) = 0 \]
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For \( T_{j,k} \) we have

\[ R_{j,k}(\infty) = 0 \]

\[
\begin{array}{ccc}
T_{11} & T_{21} & T_{2,2} \\
T_{31} & T_{3,2} & T_{3,3}
\end{array}
\]

\[
\begin{array}{ccc}
& & \text{90°} \\
& \text{90°} & \text{90°} \\
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\[ y' = \lambda(y - \varphi(t)) + \varphi'(t), \quad \Re(\lambda) < 0, \quad y(t_0) = \varphi(t_0) \]

solution: \[ y(t) = \varphi(t) \]
Stabilities and order reduction

Prothero & Robinson Test Equation

$\lambda = -10^6$, $T_{22}$

$|_{\varphi(t_0 + h_j)} - y_{h_j}|$

$O(h^2)$

$O(h^3)$
Stabilities and order reduction

Prothero & Robinson Test Equation

$\lambda : -10^6, T_{33}$

<table>
<thead>
<tr>
<th>$\varphi(t_0 + h) \text{ vs } y_{ij}$</th>
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<tbody>
<tr>
<td>$h$</td>
</tr>
<tr>
<td>$10^{-1}$</td>
</tr>
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<td>$5.0 \times 10^{-1}$</td>
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- order reduction: $O(h^3)$
- consistency order: $O(h^4)$
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modification renders extrapolation \textit{stiffly accurate}
Order reduction – the transport problem

**Transport**, adaption from [HV03]

\[ ut + au_x = 0 \]

for \( 0 \leq t, x \leq 1, a > 0 \) and

- initial function: \( u(x, 0) := u_0(x) \) with \( u_0 \in \Pi_p \)
- inflow Dirichlet condition: \( u(0, t) := u_0(-at) \)
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**Approximation (MOL-ansatz, 1st order upwind, spatial increment \(\delta x\)):**

\[ w'(t) = Aw(t) + g(t) \]

where

\[ A = -\frac{a}{\delta x} \begin{pmatrix} 1 & 1 & & \cdots & 1 \\ -1 & 1 & -1 & \cdots & 1 \\ & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 1 \end{pmatrix} \]

and

\[ g(t) = \frac{a}{\delta x} \begin{pmatrix} u_0(-at) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \]
Transport – solution

\[ a = 2, u_0(x) = x^3 + x^2 + x + 1 \]
Transport – expected behaviour

\[ a = 2, \quad u_0(x) = x^3 + x^2 + x + 1, \quad T = 1 \]
Transport – order reduction

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From the discussion in [HV03] it follows that the *stage order* (in breve StO) rules the error behaviour.
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- For the extrapolation scheme the stages are represented by the linearly implicit Euler substeps:

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  $$\rightarrow \text{for any extrapolation degree } k \text{ it holds that } \text{StO} = 1$$
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- For the extrapolation scheme the stages are represented by the linearly implicit Euler substeps:

  \[ \text{order: 1, 2, 3} \]

\[ \rightarrow \text{for any extrapolation degree } k \text{ it holds that StO = 1} \]

- In general for classical Rosenbrock/W-methods the upper bound depends on the related fully implicit method:

  \( \text{(S)DIRK: StO} = 1, \quad \text{ESDIRK: StO} \leq 2. \)

Thus, in any case StO \( \leq 2. \)
Establishing stage order – Block-Diagonal RK methods

In [BC90] Butcher and Cash consider fully implicit RK methods

\[ y_i = y_0 + \sum_{j=1}^{s} a_{ij} k_j \]

where

\[ k_i = h^\gamma f(t_0 + c_i h^\gamma, y_0 + \sum_{j=1}^{s} a_{ij} k_j) \]
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with \( h_\gamma := \gamma h, \gamma^{-1} = \max_i c_i = c_s \) and a Butcher tableau of the form

\[
\begin{array}{c|ccc}
  c_1 & a_{11} & \cdots & a_{1p} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_p & a_{p1} & \cdots & a_{pp} \\
  c_{p+1} & a_{p+1,1} & \cdots & a_{p+1,p} & 1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  c_s & a_{s1} & \cdots & a_{sp} & \cdots & a_{s,s-1} & 1 \\
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StO-condition \( C'(\eta) \): \[
\sum_{j=1}^{s} a_{ij} c_j^{q-1} = \frac{c_i^q}{q}, \quad i = 1, \ldots, s, \quad q = 1, \ldots, \eta
\]
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\quad & a_{s1} & \cdots & a_{sp} & \cdots & a_{s,s-1} & 1 & 1 \\
\end{array}
\]

by design

\[ R_i(\infty) = 0 \]

\[ \delta_{i,h}^{PR} \in O\left(\frac{h^{p+1}}{z}\right) \]

StO-condition \( C(\eta) : \quad \sum_{j=1}^{s} a_{ij} c_j^{q-1} = \frac{c_i^q}{q}, \quad i = 1, \ldots, s, \quad q = 1, \ldots, \eta \)
Achieved order – the fully implicit part

**Theorem (Order)**

Consider a Block-Diagonal RK method and let $C(\eta)$ hold for $\eta = p$ then

$$y(t_0 + c_i h) - y_i \in O(h^{p+1}) \quad \text{for} \quad i = 1, \ldots, s,$$

which means that the stage values $y_i$ are of order $p$. If additionally

$$\sum_{j=1}^{s} a_{sj} c_j^p = \frac{c_s^{p+1}}{p + 1}$$

is true then we even have

$$y(t_0 + h) - y_s \in O(h^{p+2}),$$

i. e., $y_s$ is an approximation of order $p + 1$. 

**Proof.**

The above follows directly from Butcher's Theorem [B64] and from $c_s h = h$. 

Tim Steinhoff — Construction of Finite Iteration Runge-Kutta Methods
Achieved order – the fully implicit part

**Theorem (Order)**

Consider a Block-Diagonal RK method and let $C(\eta)$ hold for $\eta = p$ then

$$y(t_0 + c_i h\gamma) - y_i \in O(h^{p+1})$$

for $i = 1, \ldots, s$,

which means that the stage values $y_i$ are of order $p$. If additionally

$$\sum_{j=1}^{s} a_{sj} c_j^p = \frac{c_s^{p+1}}{p + 1}$$

is true then we even have

$$y(t_0 + h) - y_s \in O(h^{p+2})$$

i. e., $y_s$ is an approximation of order $p + 1$.

**Proof.**

The above follows directly from Butcher’s Theorem [B64] and from $c_s h\gamma = h$. □
Division of labour

Let

\[ Y_h := (y(t_0 + c_1 h \gamma)^T, \ldots, y(t_0 + c_{s-1} h \gamma)^T, y(t_0 + h)^T) \]

\[ K := (k_1^T, \ldots, k_s^T) \]

\[ A := (a_{ij})_{i,j=1,\ldots,s}, \quad a_i^T := i\text{-th row of } A, \quad e := (1, \ldots, 1) \in \mathbb{R}^s \]
Division of labour

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\[ A := (a_{ij})_{i,j=1,\ldots,s}, \quad a_i^T := i\text{-th row of } A, \quad e := (1, \ldots, 1) \in \mathbb{R}^s \]

By means of the Order Theorem it holds that \((i = 1, \ldots, s):\)

\[ y(t_0 + c_i h \gamma) - y_i \in O(h^{p+1}) \]
\[ y(t_0 + h) - y_s \in O(h^{p+2}) \]

(covered by Butcher’s work)
Division of labour

Let

\[ Y_h := (y(t_0 + c_1 h \gamma)^T, \ldots, y(t_0 + c_{s-1} h \gamma)^T, y(t_0 + h)^T) \]
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\[ y(t_0 + c_i h \gamma) - y_i \in O(h^{p+1}) \]
\[ y(t_0 + h) - y_s \in O(h^{p+2}) \]

\[ \iff \]

\[ (Y_h - e \otimes y_0)^{(q)}|_{h=0} - (A \otimes I)K^{(q)}|_{h=0} = 0, \quad q = 0, \ldots, p \]
\[ (y(t_0 + h) - y_0)^{(p+1)}|_{h=0} - (a_s^T \otimes I)K^{(p+1)}|_{h=0} = 0 \]
Division of labour, ct. – basic approximation idea

Find cheaply computable $\tilde{K} : \mathbb{R} \supset D_h \to \mathbb{R}^{s \cdot n}$ with

$$(A \otimes I)\tilde{K}^{(q)}|_{h=0} = (A \otimes I)K^{(q)}|_{h=0}$$

$A$ nonsingular $\iff$

$\tilde{K}^{(q)}|_{h=0} = K^{(q)}|_{h=0}$

for $q = 0, \ldots, p$ and

$$(a^T_s \otimes I)\tilde{K}^{(p+1)}|_{h=0} = (a^T_s \otimes I)K^{(p+1)}|_{h=0}.$$
Division of labour, ct. – basic approximation idea

Find cheaply computable \( \tilde{K} : \mathbb{R} \supset D_h \rightarrow \mathbb{R}^{s \cdot n} \) with

\[
(\mathcal{A} \otimes I) \tilde{K}^{(q)}_{|h=0} = (\mathcal{A} \otimes I) K^{(q)}_{|h=0}
\]

\( \mathcal{A} \) nonsingular \( \iff \)

\[
\tilde{K}^{(q)}_{|h=0} = K^{(q)}_{|h=0}
\]

for \( q = 0, \ldots, p \) and

\[
(a_s^T \otimes I) \tilde{K}^{(p+1)}_{|h=0} = (a_s^T \otimes I) K^{(p+1)}_{|h=0}.
\]

Change to autonomous system for the upcoming analysis:

\[
y \rightarrow \begin{pmatrix} t \\ y \end{pmatrix}, \quad f \rightarrow \begin{pmatrix} 1 \\ f \end{pmatrix}, \quad k_i \rightarrow \begin{pmatrix} 1 \\ k_i \end{pmatrix}
\]

For the sake of simplicity stay with the notation \( y, f, k_i, K, n \) and work with \( h \), i. e.,

\[
\text{coeff} \cdot h \gamma = (\text{coeff} \cdot \gamma) \cdot h.
\]
The block stages

Let $K_b := (k_1^T, \ldots, k_p^T)$ and

$$F(\kappa) = (F_i(\kappa))_{i=1,\ldots,p}, \quad F_i(\kappa) := f(y_0 + (a_{i,1:p}^T \otimes I)\kappa).$$

Consider one step of a Newton-like iteration for the block:

$$K_b^{l+1} = hF(K_b^l) + h(I \otimes J)(A_b \otimes I)(K_b^{l+1} - K_b^l). \tag{1}$$
The block stages

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(1)

**Theorem (Block)**

Assume that for the initial guess $K_b^0 : \mathbb{R} \ni D_h \rightarrow \mathbb{R}^{p \cdot n}$ it holds that

$$K_b^0(r)|_{h=0} = K_b^{(r)}|_{h=0}, \quad r = 0, \ldots, \varrho$$

and define $K_b^l$ for $l \geq 1$ by (1). Then

$$K_b^l(q)|_{h=0} = K_b^{(q)}|_{h=0}, \quad q = 0, \ldots, \varrho + l.$$
The block stages

Let \( K_b := (k_1^T, \ldots, k_p^T) \) and

\[
F(\kappa) = (F_i(\kappa))_{i=1,\ldots,p}, \quad F_i(\kappa) := f(y_0 + (a_{i,1:p}^T \otimes I)\kappa).
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\[
K_b^l(q)|_{h=0} = K_b(q)|_{h=0}, \quad q = 0, \ldots, \varrho + l.
\]

Proof by nested induction applying Leibniz’ formula: \( (h\psi)^{(q)}|_{h=0} = q \cdot (\psi(h))^{(q-1)}|_{h=0} \).
The diagonal stages

Consider one step of a Newton-like iteration for diagonal stage $i$, $i \in \{p+1, \ldots, s\}$:

\[
k_i^{l+1} = h f \left( y_0 + \sum_{j=1}^{i-1} a_{ij} k_j^l + a_{ii} k_i^l \right) + h a_{ii} J (k_i^{l+1} - k_i^l)
\]

where the $k_j^l$'s are the final approximations from the prior stages.
The diagonal stages

Consider one step of a Newton-like iteration for diagonal stage \( i, i \in \{ p+1, \ldots, s \} \):

\[
k_i^{l+1} = h f \left( y_0 + \sum_{j=1}^{i-1} a_{ij} k_j^l + a_{ii} k_i^l \right) + h a_{ii} J (k_i^{l+1} - k_i^l)
\]

where the \( k_j^l \)'s are the final approximations from the prior stages. Exploit these for the initial guess \( k_i^0 \):

\[
k_i^0 := -\gamma^{-1}_{ii} \sum_{j=1}^{i-1} \gamma_{ij} k_j^l \quad \text{where} \quad \gamma_{ii} := a_{ii},
\]

i. e. (with \( \alpha_{ij} + \gamma_{ij} = a_{ij} \)):

\[
k_i^1 = h f \left( y_0 + \sum_{j=1}^{i-1} \alpha_{ij} k_j^l \right) + h J \left( \sum_{j=1}^{i-1} \gamma_{ij} k_j^l + \gamma_{ii} k_i^1 \right).
\]
The diagonal stages, ct.

Newton-like iteration for diagonal stage $i$:

$$k_i^{l+1} = hf(y_0 + \sum_{j=1}^{i-1} a_{ij} k_j^l + a_{ii} k_i^l) + h a_{ii} J(k_i^{l+1} - k_i^l)$$  \hspace{1cm} (2a)

$$l = 0: \quad k_i^1 = hf(y_0 + \sum_{j=1}^{i-1} \alpha_{ij} k_j^l) + h J(\sum_{j=1}^{i-1} \gamma_{ij} k_j^l + \gamma_{ii} k_i^1)$$  \hspace{1cm} (2b)
The diagonal stages, ct.

Newton-like iteration for diagonal stage $i$:

\begin{align}
    k_{i}^{l+1} &= hf(y_0 + \sum_{j=1}^{i-1} a_{ij} k_{j}^{l} + a_{ii} k_{i}^{l}) + h a_{ii} J(k_{i}^{l+1} - k_{i}^{l}) \\
    l &= 0: \quad k_{i}^{1} = hf(y_0 + \sum_{j=1}^{i-1} \alpha_{ij} k_{j}^{l}) + h J(\sum_{j=1}^{i-1} \gamma_{ij} k_{j}^{l} + \gamma_{ii} k_{i}^{1})
\end{align}

(2a) (2b)

Theorem (Diagonal stages)

Assume that

\[ k_{j}^{l}(m)_{|h=0} = k_{j}^{(m)}_{|h=0}, \quad m = 0, \ldots, \mu, \quad j = 1, \ldots, i - 1 \]

\[ \sum_{j=1}^{i-1} \gamma_{ij} k_{j}^{l}(v)_{|h=0} + \gamma_{ii} k_{i}^{l}(v)_{|h=0} = 0, \quad v = 0, \ldots, \varphi, \]

holds and define $k_{i}^{l}$ for $l \geq 1$ by (2). Then

\[ k_{i}^{l}(q)_{|h=0} = k_{i}^{(q)}_{|h=0}, \quad q = 0, \ldots, \min(\mu + 1, \varphi + l). \]
The diagonal stages, ct.

Newton-like iteration for diagonal stage $i$:

\[
k_{i}^{l+1} = hf(y_0 + \sum_{j=1}^{i-1} a_{ij} k_{j}^{l_j} + a_{ii} k_{i}^{l_i}) + ha_{ii} J(k_{i}^{l+1} - k_{i}^{l}) \tag{2a}
\]

\[
l = 0: \quad k_{i}^{1} = hf(y_0 + \sum_{j=1}^{i-1} \alpha_{ij} k_{j}^{l_j}) + hJ(\sum_{j=1}^{i-1} \gamma_{ij} k_{j}^{l_j} + \gamma_{ii} k_{i}^{1}) \tag{2b}
\]

Theorem (Diagonal stages)

Assume that

\[
k_{j}^{l_j(m)} \big|_{h=0} = k_{j}^{(m)} \big|_{h=0}, \quad m = 0, \ldots, \mu, \quad j = 1, \ldots, i - 1
\]

\[
\sum_{j=1}^{i-1} \gamma_{ij} k_{j}^{l_j(v)} \big|_{h=0} + \gamma_{ii} k_{i}^{(v)} \big|_{h=0} = 0, \quad v = 0, \ldots, \varphi,
\]

holds and define $k_{i}^{l}$ for $l \geq 1$ by (2). Then

\[
k_{i}^{l(q)} \big|_{h=0} = k_{i}^{(q)} \big|_{h=0}, \quad q = 0, \ldots, \min(\mu + 1, \varphi + l).
\]

Again, proof by nested induction applying Leibniz' formula: \((h\psi)^{(q)} \big|_{h=0} = q \cdot (\psi(h))^{(q-1)} \big|_{h=0}\)
Algebraic conditions on $\alpha_{ij}$ and $\gamma_{ij}$
Theorem (Condensation)

Let $q \geq 1$ and assume that $C(\eta)$ holds for $\eta \geq q - 1$ then there exists

$$\phi_q = \phi_q \left( f^{(0)}(y_0), \ldots, f^{(q-1)}(y_0) \right) \in \mathbb{R}^n$$

such that

$$k_i^{(q)} \bigg|_{h=0} = \phi_q \left( f^{(0)}(y_0), \ldots, f^{(q-1)}(y_0) \right) \cdot c_i^{q-1}, \quad i = 1, \ldots, s,$$

and $\phi_q$ is independent of $i$. 
Algebraic conditions on $\alpha_{ij}$ and $\gamma_{ij}$

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Let $q \geq 1$ and assume that $C(\eta)$ holds for $\eta \geq q - 1$ then there exists $\phi_q = \phi_q\left(f^{(0)}(y_0), \ldots, f^{(q-1)}(y_0)\right) \in \mathbb{R}^n$ such that

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and $\phi_q$ is independent of $i$.

The proof is by induction, technical and makes use of Faà di Bruno’s formula.
Algebraic conditions on $\alpha_{ij}$ and $\gamma_{ij}$

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$$

and $\phi_q$ is independent of $i$.

The proof is by induction, technical and makes use of Faà di Bruno’s formula.

**Faà di Bruno’s formula** – generalizing the chain rule to higher derivatives

$$
(f \circ g)^{(n)} = \sum_{(m_1, \ldots, m_n) \in T_n} \frac{n!}{m_1! \cdots m_n!} (f \circ g)^{(m_1+\cdots+m_n)} \prod_{j=1}^{n} \left( \frac{g^{(j)}}{j!} \right)^{m_j}
$$

where

$$
T_n := \left\{ (m_1, \ldots, m_n) \in \mathbb{N} \cup \{0\} \mid \sum_{i=1}^{n} i \cdot m_i = n \right\}
$$
Algebraic conditions on $\alpha_{ij}$ and $\gamma_{ij}$

**Theorem (Condensation)**

Let $q \geq 1$ and assume that $C(\eta)$ holds for $\eta \geq q - 1$ then there exists

$$\phi_q = \phi_q(f^{0}(y_0), \ldots, f^{(q-1)}(y_0)) \in \mathbb{R}^n$$

such that

$$k^{(q)}_i |_{h=0} = \phi_q(f^{0}(y_0), \ldots, f^{(q-1)}(y_0)) \cdot c^{q-1}, \quad i = 1, \ldots, s,$$

and $\phi_q$ is independent of $i$.

Assume that $C(\eta)$ with $\eta = p$ holds and that we have by the Block Theorem

$$K^{(q)}_b |_{h=0} = K^{(q)}_b |_{h=0}, \quad q = 0, \ldots, p.$$
Algebraic conditions on $\alpha_{ij}$ and $\gamma_{ij}$

**Theorem (Condensation)**

Let $q \geq 1$ and assume that $C(\eta)$ holds for $\eta \geq q - 1$ then there exists $\phi_q = \phi_q(f^{(0)}(y_0), \ldots, f^{(q-1)}(y_0)) \in \mathbb{R}^n$ such that

$$k_{i\mid h=0}^{(q)} = \phi_q(f^{(0)}(y_0), \ldots, f^{(q-1)}(y_0)) \cdot c_i^{q-1}, \quad i = 1, \ldots, s,$$

and $\phi_q$ is independent of $i$.

Assume that $C(\eta)$ with $\eta = p$ holds and that we have by the Block Theorem

$$K_b^{(q)}_{\mid h=0} = K_b^{(q)}_{\mid h=0}, \quad q = 0, \ldots, p.$$

Then for $i = p + 1$ (first D-stage), $q \in \{0, \ldots, p\}$, and assuming $\phi_q \neq 0$ we have

$$\sum_{j=1}^{i-1} \gamma_{ij} k_{j\mid h=0}^{(q)} + \gamma_{ii} k_{i\mid h=0}^{(q)} \overset{!}{=} 0 \quad \text{CT} \quad \sum_{j=1}^{i} \gamma_{ij} c_j^{q-1} \overset{!}{=} 0$$

$$C(p), a = \alpha + \gamma \quad \sum_{j=1}^{i-1} \alpha_{ij} c_j^{q-1} \overset{!}{=} c_i^q / q.$$
Algebraic conditions on $\alpha_{ij}$ and $\gamma_{ij}$, ct.

**Theorem ($\alpha$-Order)**

Assume that $C(\eta)$ with $\eta = p$ holds and that

$$K_b^{l_b(q)}|_{h=0} = K_b^{(q)}|_{h=0}, \quad q = 0, \ldots, p,$$

is true. Define $\alpha_{ij}$ and $\gamma_{ij}$ by the (affine) linear conditions

$$\alpha\text{-StO } C_\alpha(p) : \quad \sum_{j=1}^{i-1} \alpha_{ij} c_j^{q-1} = \frac{c_i^q}{q} \quad \text{and} \quad \gamma_{ij} = a_{ij} - \alpha_{ij} \quad (\alpha_{ii} := 0)$$

for $q = 1, \ldots, p$. Then,

$$k_i^{(q)}|_{h=0} = k_i^{(q)}|_{h=0}, \quad i = p + 1, \ldots, s,$$

$$q = 0, \ldots, p + 1,$$

and (3) and (4) imply that all stages of the related FlterRK method are of order $p$. 
Algebraic conditions on $\alpha_{ij}$ and $\gamma_{ij}$, ct.

**Theorem ($\alpha$-Order)**

Assume that $C(\eta)$ with $\eta = p$ holds and that

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$$\alpha - \text{StO} \quad C_{\alpha}(p) : \quad \sum_{j=1}^{i-1} \alpha_{ij} c_j^{q-1} = \frac{c_i^q}{q} \quad \text{and} \quad \gamma_{ij} = a_{ij} - \alpha_{ij} \quad (\alpha_{ii} := 0)$$

for $q = 1, \ldots, p$. Then,

$$k_i^{1(q)}|_{h=0} = k_i^{(q)}|_{h=0}, \quad i = p + 1, \ldots, s, \quad q = 0, \ldots, p + 1,$$

and (3) and (4) imply that all stages of the related FlterRK method are of order $p$.

What about order $p + 1$ for the last stage?
The T2-update

- use non-autonomous notation and exact $t$-information, i.e., $f(t_0 + c_i h, \cdot)$
- Set $K_b^0 = 0$ and change the variables $Z_b := (A_b \otimes I)K_b \ (\approx Y_b - e_b \otimes y_0)$
The T2-update

- use non-autonomous notation and exact $t$-information, i.e., $f(t_0 + c_i h, \cdot)$
- Set $K^0_b = 0$ and change the variables $Z_b := (A_b \otimes I)K_b \ (\approx Y_b - e_b \otimes y_0)$

Consider the first step of a modified Newton-like iteration for the block ($Z^0_b = 0$):

$$[I - hA_b \otimes (J + uv^T)] Z^1_b = h(A_b \otimes I) \begin{pmatrix} f(t_0 + c_1 h, y_0) \\ \vdots \\ f(t_0 + c_p h, y_0) \end{pmatrix}, \quad u, v \in \mathbb{R}^n$$
The T2-update

- use non-autonomous notation and exact $t$-information, i.e., $f(t_0 + c_i h, \cdot)$
- Set $K^0_b = 0$ and change the variables $Z_b := (A_b \otimes I)K_b \approx (Y - e_b \otimes y_0)$

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Define $f_0 := f(t_0, y_0)$, $f_0^{(1)} := f^{(1)}(t_0, y_0)$ and let $C_b := \text{diag}(c_1, \ldots, c_p)$.

Recall that $C(\eta)$ with $\eta = p$ holds and let $p \geq 2$. Then,

$$Z^{(q)}_b |_{h=0} = Z^{(q)}_b |_{h=0}, \quad q = 0, 1,$$

(by the Block Theorem)

$$Z^{(2)}_b |_{h=0} = C^2_b e_b \otimes \left( \frac{\partial f}{\partial t} (t_0, y_0) J + uv^T \right) \begin{pmatrix} 1 \\ f_0 \end{pmatrix},$$

$$Z^{(2)}_b |_{h=0} = C^2_b e_b \otimes \left( \frac{\partial f}{\partial t} (t_0, y_0) f_0^{(1)} \right) \begin{pmatrix} 1 \\ f_0 \end{pmatrix}.$$
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Define $f_0 := f(t_0, y_0)$, $f^{(1)}_0 := f^{(1)}(t_0, y_0)$ and let $C_b := \text{diag}(c_1, \ldots, c_p)$.

Recall that $C(\eta)$ with $\eta = p$ holds and let $p \geq 2$. Then,

$$Z^1_b(q) \mid_{h=0} = Z_b(q) \mid_{h=0}, \quad q = 0, 1, \quad \text{(by the Block Theorem)}$$

$$Z^1_b(2) \mid_{h=0} = C^2_b e_b \otimes \left( \frac{\partial f}{\partial t}(t_0, y_0) J + uv^T \right) \begin{pmatrix} 1 \\ f_0 \end{pmatrix},$$

$$Z_b(2) \mid_{h=0} = C^2_b e_b \otimes \left( \frac{\partial f}{\partial t}(t_0, y_0) f^{(1)}_0 \right) \begin{pmatrix} 1 \\ f_0 \end{pmatrix}.$$  

If $J = f^{(1)}_0$ or $f_0 = 0$ set $u = v = 0$: 2nd order information for free!
The T2-update & handling of the block

If \( J \neq f_{0}^{(1)} \) and \( f_{0} \neq 0 \) define \( f_{0}^{n} := f_{0}/\|f_{0}\|_{2} \) and set

\[
    u = (f_{0}^{(1)} - J)f_{0}^{n}, \quad v = f_{0}^{n}.
\]

(T2-update)

Then,

\[
    Z_{b}^{1(q)}|_{h=0} = Z_{b}^{(q)}|_{h=0}, \quad q = 0, 1, 2.
\]
The T2-update & handling of the block

If \( J \neq f_0^{(1)} \) and \( f_0 \neq 0 \) define \( f_0^n := f_0 / \| f_0 \|_2 \) and set

\[
    u = (f_0^{(1)} - J) f_0^n, \quad v = f_0^n. \tag{T2-update}
\]

Then,

\[
    Z_1^b(q) \big|_{h=0} = Z_b^{(q)}(q), \quad q = 0, 1, 2.
\]

The T2-update has a least-change property

\[
    \arg\min_{B \in \Theta} \| B \|_2 = uv^T
\]

where

\[
    \Theta := \{ B \in \mathbb{R}^{n \times n} \mid (J + B) f_0 = f_0^{(1)} f_0 \}.
\]
The T2-update & handling of the block

If \( J \neq f_0^{(1)} \) and \( f_0 \neq 0 \) define \( f_0^n := f_0 / \| f_0 \|_2 \) and set

\[
\begin{align*}
    u &= (f_0^{(1)} - J) f_0^n, \\
v &= f_0^n.
\end{align*}
\]

(T2-update)

Then,

\[
Z_b^{(q)} \big|_{h=0} = Z_b^{(q)} , \quad q = 0, 1, 2.
\]

Note that,

\[
\begin{align*}
    Z_1^{(3)} \big|_{h=0} &= A_b C_b^2 e_b \otimes \left( (J + uu^T)^2 f_0 + \partial^2 f_0 / \partial t^2 \right) \cdot 3 \\
    Z_{(3)} \big|_{h=0} &= A_b C_b^2 e_b \otimes \phi_3 \left( \partial^{(1,2)} f_0 / \partial t^{(1,2)} , f_0^{(0,1,2)} \right).
\end{align*}
\]
The T2-update & handling of the block

If $J \neq f_0^{(1)}$ and $f_0 \neq 0$ define $f_0^n := f_0/\|f_0\|_2$ and set

$$u = (f_0^{(1)} - J)f_0^n, \quad v = f_0^n.$$  \hfill (T2-update)

Then,

$$Z_{b|h=0}^{1(q)} = Z_{b|h=0}^{(q)}, \quad q = 0, 1, 2.$$

Note that,

$$Z_{|h=0}^{1(3)} = A_b C_b^2 e_b \otimes ((J + uv^T)^2 f_0 + \partial^2 f_0/\partial t^2) \cdot 3$$

$$Z_{|h=0}^{(3)} = A_b C_b^2 e_b \otimes \phi_3(\partial^{(1,2)} f_0/\partial t^{(1,2)}, f_0^{(0,1,2)}).$$

Let $a_{s,1:p} := (a_{s1}, \ldots, a_{sp})$:

$$p = 2 : (a_{s,1:p}^T \otimes I) K_{b|h=0}^{1(3)} = a_{s,1:p}^T C_b^2 e_b \otimes (\cdots) \text{ has impact} \rightarrow a_{s,1:p}^T C_b^2 e_b \equiv 0$$

$$p = 3 : (a_{s,1:p}^T \otimes I) Z_{|h=0}^{1(3)} = a_{s,1:p}^T C_b^3 e_b \otimes (\cdots) \text{ has impact} \rightarrow a_{s,1:p}^T C_b^3 e_b \equiv 0$$
Handling of the block, ct.

For the general case we obtain

**Theorem (T2 & Cancellation)**

If \( f_0 \neq 0 \) choose \( u, v \) according to the T2-update, otherwise let \( u = v = 0 \). Set \( K_0^b = 0 \) and perform \( p - 1 \) modified Newton-like iteration steps. Assume that the cancellation condition

\[
a_{s,1:p}^T C_b^p e_b = 0
\]

holds. Then,

\[
K_b^{p-1(q)} |_{h=0} = K_b^{(q)} |_{h=0}, \quad q = 0, \ldots, p,
\]

and

\[
(a_{s,1:p}^T \otimes I) K_b^{p-1(p+1)} |_{h=0} = (a_{s,1:p}^T \otimes I) K_b^{(p+1)} |_{h=0}.
\]
Handling of the block, ct.

For the general case we obtain

**Theorem (T2 & Cancellation)**

If $f_0 \neq 0$ choose $u, v$ according to the T2-update, otherwise let $u = v = 0$. Set $K^0_b = 0$ and perform $p - 1$ modified Newton-like iteration steps. Assume that the cancellation condition

$$a^{T}_{s,1:p} C^p_b e_b = 0$$

holds. Then,

$$K^{p-1(q)}_b |_{h=0} = K^{(q)}_b |_{h=0}, \quad q = 0, \ldots, p,$$

and

$$(a^{T}_{s,1:p} \otimes I) K^{p-1(p+1)}_b |_{h=0} = (a^{T}_{s,1:p} \otimes I) K^{(p+1)}_b |_{h=0}.$$ 

The T2-update saves one block-Newton-like step.
Handling of the block, ct.

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holds. Then,

$$K^{p-1(q)}_b|_{h=0} = K^{(q)}_b|_{h=0}, \quad q = 0, \ldots, p,$$

and

$$(a^{T}_{s,1:p} \otimes I) K^{p-1(p+1)}_b|_{h=0} = (a^{T}_{s,1:p} \otimes I) K^{(p+1)}_b|_{h=0}.$$

The T2-update saves one block-Newton-like step. However, an additional single Newton-like step will be introduced.
Making a block viable, basic idea cited in [HW10]

Consider step \(l\) of the modified Newton-like iteration for the block

\[
[I - hA_b \otimes (J + uv^T)] \Delta Z^l_b = -Z^l_b + h(A_b \otimes I) \bar{F}(Z^l_b)
\]

where \(\bar{F}(Z) = (\bar{F}_i(Z_i))_{i=1,...,p}, \bar{F}_i(Z_i) := f(t_0 + c_i h, y_0 + Z_i), Z = (Z^T_1, \ldots, Z^T_p)\).
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Consider step $l$ of the modified Newton-like iteration for the block

$$[I - h A_b \otimes (J + uv^T)] \Delta Z^l_b = -Z^l_b + h(A_b \otimes I)\bar{F}(Z^l_b)$$  \hspace{1cm} (5)

where $\bar{F}(Z) = (\bar{F}_i(Z_i))_{i=1,...,p}$, $\bar{F}_i(Z_i) := f(t_0 + c_i h, y_0 + Z_i)$, $Z = (Z_T^1, \ldots, Z_T^p)$.

$$\text{spec}(A_b) = \{\gamma > 0\} \implies A_b^{-1} = T \Lambda T^{-1} \quad \Lambda \text{ lower left triangular and } (\Lambda)_{ii} = \gamma^{-1}$$

$T$ nonsingular

\[\lambda\]
Making a block viable, basic idea cited in [HW10]

Consider step $l$ of the modified Newton-like iteration for the block

$$ [I - hA_b \otimes (J + uv^T)] \Delta Z_b^l = -Z_b^l + h(A_b \otimes I)\bar{F}(Z_b^l) $$  \hspace{1cm} (5)

where $\bar{F}(Z) = (\bar{F}_i(Z_i))_{i=1,\ldots,p}$, $\bar{F}_i(Z_i) := f(t_0+c_ih, y_0+Z_i)$, $Z = (Z_1^T, \ldots, Z_p^T)$.

\[ \text{spec}(A_b) = \{ \gamma > 0 \} \implies A_b^{-1} = T\Lambda T^{-1} \quad \Lambda \text{ lower left triangular and } (\Lambda)_{ii} = \gamma^{-1} \]

Left-multiply (5) by $(hA_b)^{-1} \otimes I$ and use $W_b^l := (T^{-1} \otimes I)Z_b^l$:

$$ [h^{-1}\Lambda \otimes I - I \otimes (J + uv^T)] \Delta W^l = G(W^l, \bar{F}) $$

where $G(W^l, \bar{F}) := -h^{-1}(\Lambda \otimes I)W^l + (T^{-1} \otimes I)\bar{F}((T \otimes I)W^l)$. 

Left-multiply (5) by $(hA_b)^{-1} \otimes I$ and use $W_b^l := (T^{-1} \otimes I)Z_b^l$: 

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Making a block viable, basic idea cited in [HW10]

Consider step $l$ of the modified Newton-like iteration for the block

$$
[I - hA_b \otimes (J + uv^T)] \Delta Z^l_b = -Z^l_b + h(A_b \otimes I)\bar{F}(Z^l_b)
$$

(5)

where $\bar{F}(Z) = (\bar{F}_i(Z_i))_{i=1,...,p}$, $\bar{F}_i(Z_i) := f(t_0+c_ih, y_0+Z_i)$, $Z = (Z^T_1, \ldots, Z^T_p)$.

$$\text{spec}(A_b) = \{\gamma > 0\} \implies A_b^{-1} = T\Lambda T^{-1} \quad \Lambda \text{ lower left triangular and } (\Lambda)_{ii} = \gamma^{-1}
$$

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$$

where $G(W^l, \bar{F}) := -h^{-1}(\Lambda \otimes I)W^l + (T^{-1} \otimes I)\bar{F}((T \otimes I)W^l)$.

$p$ small systems to be solved with

$$
h^{-1}\lambda I - J - uv^T = \begin{cases} h^{-1}\lambda I - J \end{cases} \begin{cases} I - M^{-1}uv^T \end{cases} =: \tilde{M} \quad \text{same as in D-stages}
$$
Making a block viable, basic idea cited in [HW10]

Consider step $l$ of the modified Newton-like iteration for the block

$$
[I - h A_b \otimes (J + uv^T)] \Delta Z^l_b = -Z^l_b + h (A_b \otimes I) \bar{F}(Z^l_b)
$$

(5)

where $\bar{F}(Z) = (\bar{F}_i(Z_i))_{i=1,...,p}$, $\bar{F}_i(Z_i) := f(t_0+c_i h, y_0+Z_i) \quad Z = (Z^T_1, \ldots, Z^T_p)$.

$\text{spec}(A_b) = \{\gamma > 0\} \implies A_b^{-1} = T \Lambda T^{-1} \quad \Lambda$ lower left triangular and $(\Lambda)_{ii} = \gamma^{-1}$

$T$ nonsingular

Left-multiply (5) by $(h A_b)^{-1} \otimes I$ and use $W^l_b := (T^{-1} \otimes I)Z^l_b$:

$$
[h^{-1} \Lambda \otimes I - I \otimes (J + uv^T)] \Delta W^l = G(W^l, \bar{F})
$$

where $G(W^l, \bar{F}) := -h^{-1}(\Lambda \otimes I)W^l + (T^{-1} \otimes I)\bar{F}((T \otimes I)W^l)$.

$p$ small systems to be solved with $h^{-1} \lambda I - J - uv^T = \begin{cases} h^{-1} \lambda I - J \\ (I - M^{-1} u v^T) \end{cases}$

same as in D-stages

Note, [D04]: $\kappa_2(\tilde{I}) \leq \frac{1 + \theta}{1 - \theta}$, $\theta := \|\tilde{u}\|_2 < 1$ assumed
Making a block viable, basic idea cited in [HW10]

Consider step $l$ of the modified Newton-like iteration for the block

$$\left[ I - hA_b \otimes (J + uv^T) \right] \Delta Z_b^l = -Z_b^l + h(A_b \otimes I)\bar{F}(Z_b^l)$$

(5)

where $\bar{F}(Z) = (\bar{F}_i(Z_i))_{i=1,...,p}$, $\bar{F}_i(Z_i) := f(t_0+c_i h, y_0+Z_i)$, $Z = (Z_1^T, \ldots, Z_p^T)$.

$\text{spec}(A_b) = \{ \gamma > 0 \} \implies A_b^{-1} = T \Lambda T^{-1}$, $\Lambda$ lower left triangular and $(\Lambda)_{ii} = \gamma^{-1}

=: \lambda$

Left-multiply (5) by $(hA_b)^{-1} \otimes I$ and use $W_b^l := (T^{-1} \otimes I)Z_b^l$:

$$\left[ h^{-1} \Lambda \otimes I - I \otimes (J + uv^T) \right] \Delta W^l = G(W^l, \bar{F})$$

where $G(W^l, \bar{F}) := -h^{-1}(\Lambda \otimes I)W^l + (T^{-1} \otimes I)\bar{F}((T \otimes I)W^l)$.

$p$ small systems to be solved with $h^{-1}\lambda I - J - uv^T = \begin{cases} h^{-1}\lambda I - J \\ I - M^{-1}uv^T \end{cases}$ same as in D-stages

$=:M$

$=:\tilde{I}$

Note, [D04]: $\kappa_2(\tilde{I}) \leq \frac{1 + \theta}{1 - \theta}, \quad \theta := \|\tilde{u}\|_2 < 1$ assumed $\implies$ quality monitor for $J$
Making a block viable, basic idea cited in [HW10]

Consider step $l$ of the modified Newton-like iteration for the block

$$
\left[ I - hA_b \otimes (J + uv^T) \right] \Delta Z^l_b = -Z^l_b + h(A_b \otimes I)\bar{F}(Z^l_b)
$$

(5)

where $\bar{F}(Z) = (\bar{F}_i(Z_i))_{i=1,\ldots,p}$, $\bar{F}_i(Z_i) := f(t_0 + c_i h, y_0 + Z_i)$, $Z = (Z^T_1, \ldots, Z^T_p)$.

$$
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$$

Left-multiply (5) by $(hA_b)^{-1} \otimes I$ and use $W^l_b := (T^{-1} \otimes I)Z^l_b$:

$$
\left[ h^{-1}\Lambda \otimes I - I \otimes (J + uv^T) \right] \Delta W^l = G(W^l, \bar{F})
$$

where $G(W^l, \bar{F}) := -h^{-1}(\Lambda \otimes I)W^l + (T^{-1} \otimes I)\bar{F}((T \otimes I)W^l)$.

$p$ small systems to be solved with

$$
\begin{aligned}
\begin{cases}
    h^{-1}\lambda I - J - uv^T = M & \text{same as in D-stages} \\
    u = (f_0^{(1)} - J)f_n & =: \tilde{u}
\end{cases}
\end{aligned}
$$

Note, [D04]: $\kappa_2(\tilde{I}) \leq \frac{1 + \theta}{1 - \theta}$, $\theta := \|\tilde{u}\|_2 < 1$ assumed $\rightarrow$ quality monitor for $J$
**Constructed FIterRK methods**

\( p = 2: \) FIterRK3/2

\[
\begin{align*}
\text{O2} & & 0 & & \gamma \\
\text{O2} & & \gamma & & \gamma \\
\text{O2} & & \gamma & & \gamma \\
\text{O3} & & \gamma & & \gamma
\end{align*}
\]

\( \text{contraction check} \)

**\( \alpha \)-Input**

\[ \Delta y^1_s = \delta_h^{est} \]

\( C(\eta), \ C_\alpha(\eta) \) for \( \eta = p \)

\( T \) from \( A_b^{-1} = T A T^{-1} \) via Schur: \( T^{-1} = T^T \) (note that \( \gamma(\lambda) = 1 \))

\( p = 3: \) FIterRK4/3

\[
\begin{align*}
\text{O3} & & \gamma & & \gamma \\
\text{O3} & & \gamma & & \gamma \\
\text{O3} & & \gamma & & \gamma \\
\text{O4} & & \gamma & & \gamma
\end{align*}
\]

\( \text{contraction check (2xBN)} \)

**\( \alpha \)-Input**

\[ \Delta y^1_s = \delta_h^{est} \]

\( \gamma(\lambda) = 1 \)
**Constructed FilterRK methods**

\( p = 2 : \text{FilterRK3}/2 \)

\[
\begin{array}{c|ccc}
O2 & \bullet & \bullet & \bullet \\
O2 & \bullet & \bullet & \gamma \\
O2 & 1 & 0 & \gamma \\
O3 & 1 & & \\
\end{array}
\]

\( \alpha \text{-Input} \rightarrow \Delta y^1_s = \delta_h^{est} \)

\( \text{contraction check} \)

\( C(\eta), \ C_\alpha(\eta) \) for \( \eta = p \)

\( T \) from \( A_b^{-1} = T \Lambda T^{-1} \) via Schur: \( T^{-1} = T^T \) (note that \( \gamma(\lambda) = 1 \))

\# linear systems of equations:

**FilterRK3/2:**

\[
1 \cdot 2 + 1 + 3 \cdot 1 = 6
\]

1xBN T2 3xD

**FilterRK4/3:**

\[
2 \cdot 3 + 1 + 4 \cdot 1 = 11
\]

2xBN T2 4xD
Design philosophy

Let \( p (=\text{StO}) \) be given:

- \( C'(\eta), C_\alpha(\eta) \) for \( \eta = p \)
- \( C_{\text{err}}^{[s]} < C_{\text{err}}^{[s-1]} \)
- Cancellation condition
- \( R(\infty) = 0 \)
- \( 0 < R(\infty) < 1 \)
- Mandatory
- \( R(x) > 0, x \in \mathbb{R}_- \) for stage \( s \)
Design philosophy

Let \( p (=\text{StO}) \) be given:

\[
C'(\eta), \ C_{\alpha}(\eta) \quad \text{for} \quad \eta = p
\]

\[
C_{\text{err}}^{[s]} < C_{\text{err}}^{[s-1]}
\]

**Cancellation condition**

\[
R(\infty) = 0
\]

\[
0 < R(\infty) < 1
\]

**Mandatory**

\[
R(x) > 0, \ x \in \mathbb{R}_-
\]

**Stage \( s \)**

- **\( A \)-stability**
  - **\( A(\alpha) \)-stability**
    - \( \alpha \) close to \( 90^\circ \)
  - \( R(x) > 0, \ x \in \mathbb{R}_- \)

- **\( A \)-stability**
  - \( \alpha_{ij}, \ \alpha_{ij} \)
    - Of moderate size
### FilterRK3/2

<table>
<thead>
<tr>
<th>$i$</th>
<th>$c_i \approx$</th>
<th>Stage</th>
<th>$A(\alpha)$</th>
<th>$R(\infty)$</th>
<th>StA</th>
<th>$R(x) &gt; 0$, $x \in \mathbb{R}_-$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.154</td>
<td>B1</td>
<td>$90^\circ$</td>
<td>$0_+$</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>2</td>
<td>0.896</td>
<td>$B2</td>
<td>D1\alpha$</td>
<td>$90^\circ$</td>
<td>$0_-$</td>
<td>✓</td>
</tr>
<tr>
<td>3</td>
<td>0.896</td>
<td>D1</td>
<td>$90^\circ$</td>
<td>$0_-$</td>
<td>✓</td>
<td>–</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>$D2\alpha$</td>
<td>$90^\circ$</td>
<td>$\approx \frac{16}{25}$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$D2</td>
<td>D3\alpha$</td>
<td>$90^\circ$</td>
<td>$0_-$</td>
<td>✓</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>D3</td>
<td>$90^\circ$</td>
<td>$0_+$</td>
<td>✓</td>
<td>✓</td>
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</table>

\[
\frac{C_{err}^{D3}}{C_{err}^{D2}} \approx 0.15
\]
# FilterRK4/3

<table>
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<th>$i$</th>
<th>$c_i \approx$</th>
<th>Stage</th>
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<th>$R(\infty)$</th>
<th>StA</th>
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<td>✓</td>
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<td>–</td>
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<tr>
<td>3</td>
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<td>✓</td>
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<td></td>
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<td>D1</td>
<td>90°</td>
<td>0$_-$</td>
<td>✓</td>
<td>–</td>
</tr>
<tr>
<td>5</td>
<td>0.711</td>
<td>D2$_\alpha$</td>
<td>90°</td>
<td>0$_+$</td>
<td>–</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td></td>
<td>D2</td>
<td>90°</td>
<td>0$_+$</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
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<td>$\approx 86.6^\circ$</td>
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<td></td>
<td>D3\mid D4$_\alpha$</td>
<td>90°</td>
<td>0$_+$</td>
<td>✓</td>
<td>✓</td>
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<tr>
<td>7</td>
<td>1</td>
<td>D4</td>
<td>90°</td>
<td>0$_+$</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

\[
\frac{C'_{D4 \text{ err}}} {C'_{D3 \text{ err}}} \approx 0.295, \quad \frac{C'_{D3 \alpha \text{ err}}} {C'_{B3 \text{ err}}} \approx 0.365, \quad \frac{C'_{D3 \text{ err}}} {C'_{B3 \text{ err}}} \approx 0.365
\]
Transport – FIterRKmethods

\[ a = 2, \quad u_0(x) = x^3 + x^2 + x + 1, \quad T = 1 \]
Transport – FilterRKmethods

\[ a = 2, \ u_0(x) = x^4 + x^3 + x^2 + x + 1, \ T = 1 \]
Transport – FilterRKmethods

\[ a = 2, \quad u_0(x) = x^2 + x + 1, \quad T = 1 \]
Transport – FIterRKmethods

\[ a = 2, \ u_0(x) = x^3 + x^2 + x + 1, \ T = 1 \]
Transport – FIterRKmethods

\[ a = 2, \quad u_0(x) = x^3 + x^2 + x + 1, \quad T = 1 \]
Transport – FilterRK methods

\[ a = 2, \ u_0(x) = x^3 + x^2 + x + 1, \ T = 1, \ \frac{|(f_0^1)_{ij} - J_{ij}|}{|(f_0^1)_{ij}|} \leq (10 \cdot \sqrt{n})^{-1}, \ T2 \]
Transport – FlIterRK methods

\[ a = 2, \quad u_0(x) = x^3 + x^2 + x + 1, \quad T = 1, \quad \frac{|(f_0^1)_{ij} - J_{ij}|}{|(f_0^1)_{ij}|} \leq (10 \cdot \sqrt{n})^{-1}, \ 1xBN \]
Transport – FIterRKmethods

\[ a = 2, \quad u_0(x) = x^3 + x^2 + x + 1, \quad T = 1, \quad \frac{|(f_0^1)_{ij} - J_{ij}|}{|(f_0^1)_{ij}|} \leq (10 \cdot \sqrt{n})^{-1}, \quad 2xBN \]
tl;dl

- We introduced the concept of Finite Iteration RK methods (FIterRK) as an extension to $W$-methods.

- We derived easy-to-handle order conditions for a FIterRK version of block-diagonal RK methods based on $C(\eta)$.

- Constructed FIterRK methods are highly stable: high stage order, $L$-stability, stiffly accurate.

- For D-stages even the $\alpha$-input is of order $p = \text{StO}$ and well up to highly stable.

- T2-update saves one block-Newton iteration and provides quality monitor for Jacobian approximations.
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We derived easy-to-handle order conditions for a FIterRK version of block-diagonal RK methods based on $C(\eta)$.

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For D-stages even the $\alpha$-input is of order $p = \text{StO}$ and well up to highly stable.

T2-update saves one block-Newton iteration and provides quality monitor for Jacobian approximations.

to do

- Endue the FIterRK methods with an adaptive step size control.
- Make the T2 update compatible with iterative linear equations solvers.
- Exploit the approximation monitor.
- Testing, testing, testing.
tl;dr

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Thank You for Your Attention!
J.C. Butcher  
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